

LOOP COPRODUCTS IN STRING TOPOLOGY AND TRIVIALITY OF HIGHER GENUS TQFT OPERATIONS

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ABSTRACT. Cohen and Godin constructed positive boundary topological quantum field theory (TQFT) structure on the homology of free loop spaces of oriented closed smooth manifolds by associating a certain operations called string operations to orientable surfaces with parametrized boundaries. We show that all TQFT string operations associated to surfaces of genus at least one vanish identically. This is a simple consequence of properties of the loop coproduct which will be discussed in detail. One interesting property is that the loop coproduct is nontrivial only on the degree d homology group of the connected component of LM consisting of contractible loops, where $d = \dim M$, with values in the degree 0 homology group of constant loops. Thus the loop coproduct behaves in a dramatically simpler way than the loop product.

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§1. Introduction and triviality of higher genus TQFT string operations

Let M be a connected closed orientable smooth manifold of dimension d , and let $LM = \text{Map}(S^1, M)$ be its free loop space of continuous maps from the circle S^1 to M . Chas and Sullivan [CS] showed that its homology $\mathbb{H}_*(LM) = H_{*+d}(LM)$ comes equipped with an associative graded commutative product of degree $-d$, and a compatible Lie bracket of degree 1. These two products together with an operator Δ of degree 1 with $\Delta^2 = 0$, coming from the natural S^1 action on LM , give $\mathbb{H}_*(LM)$ the structure of a Batalin-Vilkovisky algebra.

The associative product called the loop product was generalized to so called string operations by Cohen and Godin [CG]. Let Σ be an orientable connected surface of genus g with p incoming and q outgoing parametrized boundary circles, where we require that $q \geq 1$. To such a surface Σ , they associated an operator μ_Σ of the form

$$\mu_\Sigma : H_*((LM)^p) \rightarrow H_{*+\chi(\Sigma)d}((LM)^q),$$

in such a way that μ_Σ depends only on the topological type of the surface Σ and μ_Σ is compatible with sewing of surfaces along parametrized boundaries. These operations give rise to topological

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quantum field theory (TQFT) without a counit. When Σ is a pair of pants with either 2 incoming or 2 outgoing circles, we get a product and a coproduct:

$$\begin{aligned}\mu &: H_*(LM \times LM) \longrightarrow H_{*-d}(LM), \\ \Psi &: H_*(LM) \longrightarrow H_{*-d}(LM \times LM),\end{aligned}$$

where the product μ coincides with the loop product of Chas and Sullivan. See formula (2-8) for a homotopy theoretic definition of the loop product. Since any surface Σ can be decomposed into pairs of pants and capping discs, we can compute the string operation μ_Σ by composing loop products and loop coproducts according to pants decompositions of Σ . In this paper, we study properties of coproduct in detail, and as a consequence we show that for higher genus surfaces Σ , the string operations μ_Σ are always trivial.

Theorem A. *Let Σ be an oriented connected compact surface of genus g with p incoming and $q \geq 1$ outgoing parametrized boundary circles. If $g \geq 1$ or $q \geq 3$, then the associated string operation μ_Σ vanishes.*

Thus the only nontrivial TQFT string operations correspond to genus 0 surfaces with at most 2 outgoing circles. To elements $a_1, a_2, \dots, a_p \in H_*(LM)$, such operations associate either their loop product $a_1 a_2 \cdots a_p$ or its loop coproduct $\Psi(a_1 a_2 \cdots a_p)$. Thus once we understand the loop coproduct Ψ , we know the behavior of all string operations associated to orientable surfaces with parametrized boundaries. For $a \in H_*(LM)$, let $|a|$ denote its homological degree. Let c_0 be the constant loop at the base point x_0 in M , and let $[c_0]$ its homology class in $H_0(LM)$.

The connected components of LM are parametrized by conjugacy classes of $\pi_1(M)$. Let $(LM)_{[1]}$ be the component corresponding to the conjugacy class of $1 \in \pi_1(M)$. This is the space of contractible loops in M .

In addition to the Frobenius compatibility (Theorem 2.2), properties of the loop coproduct are described in Theorem B, whose part (2) shows dramatic simplicity of the loop coproduct compared with the loop product. Theorem B is the main result of this paper. Theorem A is only one of the consequences of Theorem B. We will discuss some of the other consequences in Theorem C.

Theorem B. (1) *Let $p \geq 0$, and let $a_1, a_2, \dots, a_p \in H_*(LM)$ be arbitrary p elements. Then the image of the loop coproduct Ψ lies in the subset $H_*(LM) \otimes H_*(LM) \subset H_*(LM \times LM)$ of cross products, and for any $0 \leq \ell \leq p$ it is given by*

$$\Psi(a_1 \cdot a_2 \cdots a_p) = \chi(M)[c_0]a_1 \cdot a_2 \cdots a_\ell \otimes [c_0]a_{\ell+1} \cdots a_p \in H_*(LM) \otimes H_*(LM),$$

where $\chi(M)$ is the Euler characteristic of M .

(2) *The loop coproduct Ψ is nontrivial only on $H_d((LM)_{[1]})$, the degree d homology group of the component of contractible loops in M . On $H_d((LM)_{[1]})$, the loop coproduct Ψ has values in the homology classes of constant loops $H_0((LM)_{[1]}) \otimes H_0((LM)_{[1]}) \cong \mathbb{Z}[c_0] \otimes [c_0]$.*

Theorem B is proved in Theorem 3.1. Note that if M has vanishing Euler characteristic, for example if M is odd dimensional, then its loop coproduct is identically 0. Before we prove the above result in §3, in §2 we will prove various general results on the loop coproduct including Frobenius compatibility (Theorem 2.2) with precise treatment of signs, and Frobenius compatibility and coderivation compatibility with respect to cap products (Theorem 2.4).

Since the proof of Theorem A is more or less straightforward, we give its proof below. This vanishing property is the basis of triviality of stable higher string operations [T2] in the context of homological conformal field theory in which homology classes of moduli spaces of Riemann surfaces give rise to string operations [G].

As a consequence of Theorem B, we obtain the following result on torsion elements proved in §3. Let $\iota : \Omega M \rightarrow LM$ be the inclusion map from the based loop space to the free loop space. Recall that the transfer map $\iota_! : H_{*+d}(LM) \rightarrow H_*(\Omega M)$ obtained by intersecting cycles with ΩM is an algebra map with respect to the loop product in $H_*(LM)$ and the Pontrjagin product in $H_*(\Omega M)$.

Theorem C. *Let M be an even dimensional manifold with $\chi(M) \neq 0$. Consider the following composition map*

$$\iota_* \circ \iota_! : H_{p+d}(LM) \xrightarrow{\iota_!} H_p(\Omega M) \xrightarrow{\iota_*} H_p(LM).$$

If $p \neq 0$, then the image of $\iota_ \iota_!$ consists of torsion elements of order a divisor of $\chi(M)$. Namely,*

$$\chi(M) \iota_* \iota_!(a) = \chi(M)[c_0] \cdot a = 0 \quad \text{if } |a| \neq d \text{ for } a \in H_*(LM).$$

Thus, rationally, the composition is a trivial map if $p \neq 0$.

See Example 3.6 for explicit examples of this fact when M is S^{2n} or $\mathbb{C}P^n$.

Since Theorem A can be quickly proved from Theorem B, we give its proof here in the remainder of this introduction.

Proof of Theorem A from Theorem B. Let $S(p, q)$ be a genus 0 surface with p incoming and q outgoing parametrized boundary circles, and let T be a torus with 1 incoming and 1 outgoing parametrized boundary circles. Then any surface Σ of genus g with p incoming boundary circles and q outgoing boundary circles can be decomposed as $S(p, 1) \# T \# \cdots \# T \# S(1, q)$, where T appears g times. Correspondingly, the associated string operation μ_Σ can be decomposed as

$$\mu_\Sigma = \mu_{S(1,q)} \circ \mu_T \circ \cdots \circ \mu_T \circ \mu_{S(p,1)}.$$

Assume $g \geq 1$. We compute μ_T using a decomposition of T into two pairs of pants corresponding to the loop coproduct and the loop product. For any $a \in H_*(LM)$,

$$\mu_T(a) = \mu \circ \Psi(a) = \mu(\chi(M)[c_0] \otimes ([c_0] \cdot a)) = (-1)^d \chi(M)([c_0] \cdot [c_0]) \cdot a.$$

Since $[c_0] \cdot [c_0] = 0 \in H_{-d}(LM)$ by dimensional reason, we have $\mu_T(a) = 0$ for all $a \in H_*(LM)$. In view of the above decomposition of μ_Σ , this proves the vanishing of string operations associated to surfaces of genus $g \geq 1$.

Next we assume $q \geq 3$. Then $\mu_{S(1,q)} = (\mu_{S(1,q-2)} \otimes 1 \otimes 1) \circ (\Psi \otimes 1) \circ \Psi$. For any $a \in H_*(LM)$,

$$(\Psi \otimes 1) \circ \Psi(a) = (\Psi \otimes 1)(\chi(M)[c_0] \otimes [c_0] \cdot a) = \chi(M)\Psi([c_0]) \otimes [c_0] \cdot a = 0,$$

since $\Psi([c_0]) = 0 \in H_{-d}(LM \times LM)$ by dimensional reason. Hence $\mu_{S(1,q)} = 0$ for $q \geq 3$. Again, in view of the above decomposition of μ_Σ , this proves $q \geq 3$ case of Theorem A. \square

In §2, we discuss general properties of the loop coproduct in detail and prove Frobenius compatibility (Theorem 2.2), a symmetry property (Proposition 2.3), and coderivation property of certain cap products (Theorem 2.4). In §3, we prove Theorem B and related results in Theorem 3.1, and deduce their consequences including Theorem C proved in Corollary 3.3 and Corollary 3.4. We also discuss torsion properties of certain loop bracket elements in Corollary 3.5, and other miscellaneous properties of image elements of the loop coproduct in Propositions 3.7 and 3.8. All homology groups in this paper have integer coefficients.

§2. The loop coproduct and its Frobenius compatibility

As before, let LM be the free loop space of continuous maps from the circle $S^1 = \mathbb{R}/\mathbb{Z}$ to a connected oriented closed smooth d -manifold M . Cohen and Jones [CJ] gave a homotopy theoretic description of the loop product. The loop coproduct can be described in a similar way, and we study its properties in this section. A description of the loop coproduct using transversal chains is given in [S].

Let $p, p' : LM \rightarrow M$ be evaluation maps given by $p(\gamma) = \gamma(0)$ and $p'(\gamma) = \gamma(\frac{1}{2})$ for $\gamma \in LM$. We consider the following diagram where $SM = (p, p')^{-1}(\phi(M))$ consists of loops γ such that $\gamma(0) = \gamma(\frac{1}{2})$, and q is the restriction of (p, p') to this subspace. Let $\iota : SM \rightarrow LM$ be the inclusion map and let $j : SM \rightarrow LM \times LM$ be given by $j(\gamma) = (\gamma_{[0, \frac{1}{2}]}, \gamma_{[\frac{1}{2}, 1]})$. The map $\phi : M \rightarrow M \times M$ is the diagonal map.

$$\begin{array}{ccccc} LM & \xleftarrow{\iota} & SM & \xrightarrow{j} & LM \times LM \\ (p, p') \downarrow & & q \downarrow & & \\ M \times M & \xleftarrow{\phi} & M & & \end{array}$$

Then the coproduct map Ψ is defined by the following composition of maps:

$$\Psi = j_* \circ \iota_! : H_{*+d}(LM) \xrightarrow{\iota_!} H_*(LM) \xrightarrow{j_*} H_*(LM \times LM),$$

where $\iota_!$ is the transfer map, also called a push-forward map, defined in the following way. Let $\pi : \nu \rightarrow \phi(M)$ be the normal bundle to $\phi(M)$ in $M \times M$ and we orient ν so that we have an oriented isomorphism $\nu \oplus T\phi(M) \cong T(M \times M)|_{\phi(M)}$. Let N be a closed tubular neighborhood of $\phi(M)$ such that $D(\nu) \cong N$, where $D(\nu)$ is the closed disc bundle. Let $c : M \times M \rightarrow N/\partial N$ be the Thom collapse map. We have the following commutative diagram:

$$\begin{array}{ccc} H^d(M \times M, M \times M - \phi(M)) & \longrightarrow & H^*(M \times M) \\ \cong \downarrow \text{excision} & & \uparrow c^* \\ H^d(N, N - \phi(M)) & \xrightarrow{\cong} & H^d(N, \partial N) \cong \tilde{H}^d(N/\partial N). \end{array}$$

Let $u' \in \tilde{H}(N/\partial N)$ be the Thom class of the normal bundle ν . Let $u'' \in H^d(M \times M, M \times M - \phi(M))$ and $u \in H^d(M \times M)$ be corresponding Thom classes. The class u is characterized by the property $u \cap [M \times M] = \phi_*([M])$. Since u comes from u'' , it is represented by a cocycle f which vanish on simplices in $M \times M$ which do not intersect with $\phi(M)$.

Let $\tilde{N} = (p, p')^{-1}(N)$ be a tubular neighborhood of SM in LM , and let $\tilde{c} : LM \rightarrow \tilde{N}/\partial\tilde{N}$ be the Thom collapse map. Let $\tilde{u}' \in \tilde{H}^d(\tilde{N}/\partial\tilde{N})$ and $\tilde{u} \in H^d(LM)$ be pull-backs of corresponding classes. We have $\tilde{u} = \tilde{c}^*(\tilde{u}')$. Let $\tilde{\pi} : \tilde{N} \rightarrow SM \subset LM \times LM$ be a projection map corresponding to π , and is given as follows. Suppose $\gamma \in \tilde{N}$ is such that $(p, p')(\gamma) = (x_1, x_2) \in N$. Let $\eta(t) = (\eta_1(t), \eta_2(t))$ be a path in N from (x_1, x_2) to $\pi(x_1, x_2) = (y, y) \in \phi(M)$ corresponding to the straight ray in the bundle ν . Then $\tilde{\pi}(\gamma) = (\eta_1^{-1} \cdot \gamma_{[0, \frac{1}{2}]} \cdot \eta_2) \cdot (\eta_2^{-1} \cdot \gamma_{[\frac{1}{2}, 1]} \cdot \eta_1) \in SM$. From this description, it is obvious that $\tilde{\pi}$ is a deformation retraction. The transfer map $\iota_!$ is defined by the following composition of maps:

$$\iota_! : \tilde{H}_{*+d}(LM) \xrightarrow{\tilde{c}_*} \tilde{H}_{*+d}(\tilde{N}/\partial\tilde{N}) \xrightarrow{\tilde{u}' \cap (\cdot)} H_*(\tilde{N}) \xrightarrow[\cong]{\tilde{\pi}_*} H_*(SM).$$

Let $s : M \rightarrow LM$ be the constant loop map given by $s(x) = c_x$, where c_x is the constant loop at $x \in M$. Since $p \circ s = 1_M$, we have $s^* \circ p^* = 1$. The transfer map $\iota_!$ has the following properties.

Proposition 2.1. (1) *The cohomology class $\tilde{u} \in H^d(LM)$ is given by $\tilde{u} = p^*(e_M)$, where $e_M \in H^d(M)$ is the Euler class of M .*

(2) *For any element $a \in H_*(LM)$,*

$$(2-1) \quad \iota_* \iota_!(a) = p^*(e_M) \cap a.$$

In particular, $\iota_ \iota_!(s_*([M])) = \chi(M)[c_0]$, where c_0 is the constant loop at the base point x_0 in M , and $\chi(M)$ is the Euler characteristic of M .*

(3) *For any $\alpha \in H^*(LM)$ and $b \in H_*(LM)$,*

$$(2-2) \quad \iota_!(\alpha \cap b) = (-1)^{d|\alpha|} \iota^*(\alpha) \cap \iota_!(b).$$

Proof. (1) Since the map $(p, p') : LM \rightarrow M \times M$ can be factored as $LM \xrightarrow{\phi} LM \times LM \xrightarrow{p \times p'} M \times M$ and p and p' are homotopic, we have $\tilde{u} = (p, p')^*(u) = \phi^* \circ (p \times p')^*(u) = \phi^* \circ (p \times p)^*(u) = p^* \circ \phi^*(u)$. Since $\phi^*(u)$ is, by definition, $(-1)^d$ times the Euler class e_M of M and the Euler class is of order 2 when d is odd, we have $(-1)^d e_M = e_M$. So we have $\tilde{u} = p^*(e_M)$.

(2) Although we can use a certain commutative diagram for a proof (see below), we first do a chain argument here in the spirit of [CS] and [S]. By barycentric subdivisions on the cycle ξ representing $a \in H_*(LM)$, we may assume that every simplex of ξ intersecting with SM is contained in $\text{Int}(\tilde{N})$. Since cohomology classes $u \in H^d(M \times M)$ and $u' \in \tilde{H}^d(N/\partial N)$ come from the class u'' in $H^d(M \times M, M \times M - \phi(M))$, they can be represented by cocycles f and f' so that f vanishes on simplices in $M \times M$ not intersecting $\phi(M)$, and f' vanishes on simplices in $N/\partial N$ not intersecting $\phi(M)$. So the cocycle $\tilde{f}' = (p, p')^\#(f')$ representing \tilde{u}' vanishes on simplices in N not intersecting with SM . Similarly, the cocycle $\tilde{f} = (p, p')^\#(f) = \tilde{c}^\#(\tilde{f}')$ representing $\tilde{u} = \tilde{c}^*(\tilde{u}')$ vanishes on simplices in LM not intersecting with SM , and has the same values as \tilde{f}' on simplices in \tilde{N} intersecting with SM . Since the cycle ξ is fine enough, the cycles $\tilde{f} \cap \xi$ and $\tilde{f}' \cap \tilde{c}^\#(\xi)$ representing $\tilde{u} \cap a$ and $\tilde{u}' \cap \tilde{c}^*(a)$ are in fact identical. Since $\iota_!(a) = \tilde{\pi}_*(\tilde{u}' \cap \tilde{c}_*(a))$ is represented by a cycle $\tilde{\pi}_\#(\tilde{f}' \cap \xi)$, and $\tilde{\pi}$ is a deformation retraction, the two cycles $\tilde{\pi}_\#(\tilde{f}' \cap \xi)$ and $\tilde{f} \cap \xi$ are homologous inside of $\text{Int} \tilde{N}$. Thus, $\iota_* \iota_!(a) = [\tilde{\pi}_\#(\tilde{f}' \cap \xi)]$ and $\tilde{u} \cap a = [\tilde{f} \cap \xi]$ represent the same homology class. Hence $\iota_* \iota_!(a) = \tilde{u} \cap a = p^*(e_M) \cap a$, by (1).

We also give a homological proof, using the following commutative diagram.

$$\begin{array}{ccccccc} H_*(LM) & \xrightarrow{\tilde{c}_*} & H_*(\tilde{N}, \partial \tilde{N}) & \xrightarrow{\tilde{u}' \cap (\cdot)} & H_{*-d}(\tilde{N}) & \xrightarrow[\cong]{\tilde{\pi}_*} & H_{*-d}(SM) \\ \parallel & & (\iota_N)_* \downarrow \cong & & (\iota_N)_* \downarrow & & \iota_* \downarrow \\ H_*(LM) & \xrightarrow{j_*} & H_*(LM, LM - SM) & \xrightarrow{\tilde{u}'' \cap (\cdot)} & H_{*-d}(LM) & \xlongequal{\quad} & H_{*-d}(LM) \end{array}$$

where $\iota_N : \tilde{N} \rightarrow LM$ is an inclusion map. Here, the class \tilde{u}'' is given by $\tilde{u}'' = (p, p')^*(u'')$, and it satisfies $\tilde{u}' = \iota_N^*(\tilde{u}'')$. Thus, for $a \in H_*(LM)$, the commutative diagram shows $\iota_* \iota_!(a) = \tilde{u}'' \cap j_*(a) = j^*(\tilde{u}'') \cap a = \tilde{u} \cap a$. The above chain argument gives geometric meaning to the commutative diagram above.

When $a = s_*([M])$, we have $\iota_* \iota_!(s_*([M])) = p^*(e_M) \cap s_*([M]) = s_*(s^* p^*(e_M) \cap [M])$. Since $p \circ s = 1$, this is equal to $s_*(\chi(M)[x_0]) = \chi(M)[c_0]$.

(3) We compute. By definition of $\iota_!$, we have

$$\iota_!(\alpha \cap b) = \tilde{\pi}_*(\tilde{u} \cap \tilde{c}_*(\alpha \cap b)) = \tilde{\pi}_*(\tilde{u}' \cap (\iota_N^*(\alpha) \cap \tilde{c}_*(b))) = (-1)^{|\alpha|d} \tilde{\pi}_*(\iota_N^*(\alpha) \cap (\tilde{u}' \cap \tilde{c}_*(b))).$$

Since $\iota_N^*(\alpha) = \tilde{\pi}^* \iota^*(\alpha)$, the last formula becomes $\iota^*(\alpha) \cap \tilde{\pi}_*(\tilde{u}' \cap \tilde{c}_*(b)) = \iota^*(\alpha) \cap \iota_!(b)$, times the sign. This completes the proof. \square

Next we recall a homotopy theoretic description of the loop product from [CJ]. We consider the following diagram, where $LM \times_M LM$ denotes the set $(p \times p)^{-1}(\phi(M))$ consisting of pairs (γ, η) of loops such that $\gamma(0) = \eta(0)$, and $\iota(\gamma, \eta)$ denotes the usual loop multiplication $\gamma \cdot \eta$.

$$\begin{array}{ccc} LM \times LM & \xleftarrow{j} & LM \times_M LM \xrightarrow{\iota} LM \\ p \times p \downarrow & & q \downarrow \\ M \times M & \xleftarrow{\phi} & M \end{array}$$

Then for $a, b \in H_*(LM)$, the loop product $a \cdot b$ is defined by

$$a \cdot b = (-1)^{d(|a|-d)} \iota_* j_!(a \times b).$$

Here, as before, the transfer map $j_!$ is defined using the Thom class $u' \in \tilde{H}^d(N/\partial N)$ and its pull back to the tubular neighborhood $\tilde{N} = (p \times p)^{-1}(N)$. The sign $(-1)^{d(|a|-d)}$ is natural since on the right hand side, the map $j_!$, which represents the content of the loop product, is in front of a , whereas on the left hand side, the dot representing the loop product is between a and b . Switching the order of $j_!$ and a yields the sign $(-1)^{d|a|}$. The sign $(-1)^d$ comes from our choice of the orientation of the normal bundle ν so that $[M] \in H_d(LM)$ acts as the unit. Note that the $|a| - d$ is the degree of a in the loop algebra $\mathbb{H}_*(LM) = H_{*+d}(LM)$.

For further discussion, we need transfer maps defined in the following general context. Let $\iota : K \rightarrow M$ be a smooth embedding of oriented closed smooth manifolds and let ν be its normal bundle oriented by $\nu \oplus \iota_*(TK) \cong TM|_{\iota(K)}$. Let u' be the Thom class of ν and let $u \in H^{d-k}(M)$ be the corresponding Thom class for the embedding ι , where d and k are dimensions of M and K . With the above choice of the orientation on ν , we have $u \cap [M] = \iota_*([K])$, which characterizes the class u . Had we oriented ν by $\iota_*(TK) \oplus \nu \cong TM|_{\iota(K)}$, then we would have obtained $u \cap [M] = (-1)^{k(d-k)} \iota_*([K])$.

Let $p : E \rightarrow M$ be a Hurewicz fibration, and let E_K be its pull-back over K via the embedding ι . Let $\iota : E_K \rightarrow E$ be the inclusion of fibrations. Proceeding as before, we can define a transfer map.

$$\iota_! : H_{*+d}(E) \rightarrow H_{*+k}(E_K), \text{ such that } \iota_* \iota_!(a) = p^*(u) \cap a \text{ for any } a \in H_*(E).$$

We remark that with the above choice of the orientation on the normal bundle ν , the transfer map between base manifolds satisfies $\iota_!([M]) = [K]$. Also, it can be verified that for a composition of smooth embeddings $K \xrightarrow{g} L \xrightarrow{f} M$ and the associated induced inclusions of fibrations $E_K \xrightarrow{g} E_L \xrightarrow{f} E$, we have $(f \circ g)_! = g_! \circ f_!$.

The loop product enjoys the Frobenius compatibility with respect to the loop coproduct, in the following sense. This is discussed in [S] from the point of view of chains. Here, we give a homotopy theoretic proof with precise determination of signs.

For $a \in H_*(LM)$ and $c \in H_*(LM \times LM)$, let $a \cdot c$ be defined by $(\iota \times 1)_* \circ (j \times 1)_!(a \times c) = (-1)^{d(|a|-d)} a \cdot c$ using the following diagram

$$\begin{array}{ccc} (LM \times LM) \times LM & \xrightarrow{j \times 1} & (LM \times_M LM) \times LM \xrightarrow{\iota \times 1} LM \times LM \\ p_1 \times p_2 \downarrow & & p_1 \downarrow \\ M \times M & \xleftarrow{\phi} & M \end{array}$$

where $p_1 \times p_2$ denotes projections from the first and second factor. If c is of the form of a cross product $b \times c$, then $a \cdot (b \times c) = (a \cdot b) \times c$. Similarly, an element $c \cdot a$ is defined by $(1 \times \iota)_*(1 \times j)_!(c \times a) = (-1)^{d(|c|-d)}c \cdot a$ using a similar diagram.

Theorem 2.2. *The loop product and the loop coproduct satisfy Frobenius compatibility, namely, for $a, b \in H_*(LM)$,*

$$(2-3) \quad \Psi(a \cdot b) = (-1)^{d(|a|-d)}a \cdot \Psi(b) = \Psi(a) \cdot b.$$

Proof. For convenience, we introduce a space $L^r M$ of continuous loops from a circle of length $r > 0$ to M . We let $L'M = L^{\frac{1}{3}}M$ and $L''M = L^{\frac{2}{3}}M$. We identify $SM \subset L^{2r}M$ with $L^r M \times_M L^r M$. We have the following commutative diagram of inclusions:

$$\begin{array}{ccccc} L'M \times L''M & \xleftarrow{1 \times \iota} & L'M \times L'M \times_M L'M & \xrightarrow{1 \times j} & L'M \times L'M \times L'M \\ j \uparrow & & j_1 = (j \times_M 1) \uparrow & & j \times 1 \uparrow \\ L'M \times_M L''M & \xleftarrow{\iota_1 = (1 \times_M \iota)} & L'M \times_M L'M \times_M L'M & \xrightarrow{j_2 = (1 \times_M j)} & L'M \times_M L'M \times L'M \\ \iota \downarrow & & \iota_2 = (\iota \times_M 1) \downarrow & & \iota \times 1 \downarrow \\ LM & \xleftarrow{\iota} & L''M \times_M L'M & \xrightarrow{j} & L''M \times L'M. \end{array}$$

The base manifolds of fibrations in the above diagram form the following diagram which we use to compute Thom classes of embeddings, which in turn are used to construct transfer maps.

$$\begin{array}{ccccc} M \times M \times M & \xleftarrow{1 \times \phi} & M \times M & \xrightarrow{1 \times \phi} & M \times M \times M \\ \phi \times 1 \uparrow & & \phi \uparrow & & \phi \times 1 \uparrow \\ M \times M & \xleftarrow{\phi} & M & \xrightarrow{\phi} & M \times M \\ \phi \times 1 \downarrow & & \phi \downarrow & & \phi \times 1 \downarrow \\ M \times M \times M & \xleftarrow{\phi_{13}} & M \times M & \xrightarrow{\phi_{13}} & M \times M \times M \end{array}$$

where $\phi_{13}(x, y) = (x, y, x)$, or $\phi_{13} = (1 \times T)(\phi \times 1)$ and $T : M \times M \rightarrow M \times M$ is the switching map. Here, for example, the fibration $p : L''M \times_M L'M \rightarrow M \times M$ is given by $p(\gamma, \eta) = (\gamma(0) = \eta(0), \gamma(\frac{1}{3}))$, and the fibration $p : L'M \times_M L''M \rightarrow M \times M$ is given by $p(\gamma, \eta) = (\gamma(0) = \eta(0), \eta(\frac{1}{3}))$.

To prove $\Psi(a \cdot b) = (-1)^{d(|a|-d)}a \cdot \Psi(b)$, we examine the following induced homology diagram with transfers in which we replaced $L'M$ and $L''M$ by their homeomorphic copy LM .

$$\begin{array}{ccccc} H_*(LM \times LM) & \xrightarrow[= (-1)^{d1 \times \iota_!}]{(1 \times \iota)_!} & H_{*-d}(LM \times LM \times_M LM) & \xrightarrow{(1 \times j)_*} & H_{*-d}(LM \times LM \times LM) \\ \tilde{j}_! = j_! \downarrow & & (j_1)_! \downarrow & & (j \times 1)_! = j_! \times 1 \downarrow \\ H_{*-d}(LM \times_M LM) & \xrightarrow{(\iota_1)_!} & H_{*-2d}(LM \times_M LM \times_M LM) & \xrightarrow{(j_2)_*} & H_{*-2d}(LM \times_M LM \times LM) \\ \iota_* \downarrow & & (\iota_2)_* \downarrow & & (\iota \times 1)_* \downarrow \\ H_{*-d}(LM) & \xrightarrow{\tilde{\iota}_! = (-1)^d \iota_!} & H_{*-2d}(LM \times_M LM) & \xrightarrow{j_*} & H_{*-2d}(LM \times LM) \end{array}$$

In the above, the transfer maps $\tilde{j}_!, \tilde{l}_!$ indicate that Thom classes used to define these transfer maps may be different in signs from Thom classes used to define transfers $\iota_!$ and $j_!$.

The top left square and the bottom right square commute because of the functorial properties of transfer maps and induced maps. We examine the commutativity of the bottom left square. Since the corresponding square of fibrations commutes, the homology square with induced maps and transfer maps commutes up to a sign. To determine this sign, for $a \in H_*(LM \times_M LM)$, we compare $\iota_*(\iota_2)_*(\iota_1)_!(a)$ and $\iota_*\tilde{l}_!\iota_*(a)$ in $H_*(LM)$. Let $u \in H^d(M \times M)$ be the Thom class for the embedding $\phi : M \rightarrow M \times M$. Then the Thom class for the embedding $\phi_{13} : M \times M \rightarrow M \times M \times M$ is given by $(-1)^d u_{13}$, where $u_{13} = (1 \times T)^*(u \times 1) = \sum_i (u'_i \times 1 \times u''_i)$ if $u = \sum_i u'_i \times u''_i$. Hence $(\iota_*\tilde{l}_!)\iota_*(a) = (-1)^d p^*(u_{13}) \cap \iota_*(a)$, where the map $p : LM \rightarrow M \times M \times M$ is a fibration given by $p(\gamma) = (\gamma(0), \gamma(\frac{1}{3}), \gamma(\frac{2}{3}))$. On the other hand, using the commutativity of the induced homology square, we have $\iota_*(\iota_2)_*(\iota_1)_!(a) = \iota_*(\iota_1)_*(\iota_1)_!(a) = \iota_*(p^*(u) \cap a)$, since the Thom class for the embedding ι_1 is $p^*(u)$. Since $u = (\phi \times 1)^*(u_{13})$, we have $p^*(u) = p^*((\phi \times 1)^*(u_{13})) = \iota^*(p^*(u_{13}))$. Hence $\iota_*(p^*(u) \cap a) = p^*(u_{13}) \cap \iota_*(a)$. Collecting our computations, we have that $\iota_*(\iota_2)_*(\iota_1)_!(a) = p^*(u_{13}) \cap \iota_*(a)$. Comparing with the formula above for $\iota_*\tilde{l}_!\iota_*(a)$, we see that the sign difference between $(\iota_2)_*(\iota_1)_!(a)$ and $\tilde{l}_!\iota_*(a)$ is given by $(-1)^d$. Hence the square commutes up to $(-1)^d$.

Similar argument shows that the top right square in the homology diagram actually commutes.

Next we examine transfer maps in the diagram. For the top horizontal left transfer $(1 \times \iota)_!$, since the Thom class of the embedding $1 \times \phi : M \times M \rightarrow M \times M \times M$ is $(-1)^d(1 \times u)$,

$$\begin{aligned} (1 \times \iota)_*(1 \times \iota)_!(a \times b) &= (-1)^d p^*(1 \times u) \cap (a \times b) = (-1)^{d+d|a|} a \times (p^*(u) \cap b) \\ &= (-1)^{d+d|a|} a \times \iota_*\iota_!(b) = (-1)^d (1 \times \iota)_*(1 \times (\iota)_!)(a \times b). \end{aligned}$$

for $a, b \in H_*(LM)$. Thus, $(1 \times \iota)_! = (-1)^d 1 \times (\iota)_!$, as indicated in the diagram. Similarly, we can verify that for the vertical top right transfer map, we have $(j \times 1)_! = j_! \times 1$. For the vertical top left transfer $\tilde{j}_!$ associated to the Thom class for the embedding $\phi \times 1 : M \times M \rightarrow M \times M \times M$ coincides with the transfer $j_!$ associated to the Thom class for the embedding $\phi : M \rightarrow M \times M$. The bottom left horizontal transfer map $\tilde{l}_!$ associated to the Thom class $(-1)^d u_{13}$ for the embedding $\phi_{13} : M \times M \rightarrow M \times M \times M$ coincides with $(-1)^d \iota_!$, where $\iota_!$ is the transfer associated to the Thom class u of the embedding $\phi : M \rightarrow M \times M$.

Hence for $a, b \in H_*(LM)$, tracing the diagram from the top left corner to the bottom right corner via bottom left corner, we get

$$j_*(\tilde{l})!\iota_*(\tilde{j})_!(a \times b) = j_*(-1)^d \iota_!((-1)^{d(|a|-d)} a \cdot b) = (-1)^{d+d(|a|-d)} \Psi(a \cdot b).$$

Following the diagram via the top right corner, we get

$$\begin{aligned} (\iota \times 1)_*(j \times 1)_!(1 \times j)_*(1 \times \iota)_!(a \times b) &= (\iota_* \times 1)(j_! \times 1)(1 \times j_*)(-1)^d (1 \times \iota_!)(a \times b) \\ &= (-1)^{d+|a|d} (\iota_* j_! \times 1)(a \times \Psi(b)) = (-1)^{d+|a|d+d(|a|-d)} a \cdot \Psi(b). \end{aligned}$$

Since the entire diagram commutes up to $(-1)^d$, we finally get $\Psi(a \cdot b) = (-1)^{d(|a|-d)} a \cdot \Psi(b)$.

To prove the other identity $\Psi(a \cdot b) = \Psi(a) \cdot b$, we consider the induced homology diagram with

transfers flowing from the bottom right corner to the top left corner given below.

$$\begin{array}{ccccc}
 H_{*-2d}(LM \times LM) & \xleftarrow{(1 \times \iota)_*} & H_{*-2d}(LM \times LM \times_M LM) & \xleftarrow[\substack{=(1 \times j)_! \\ =(-1)^d(1 \times j_!)}]{(1 \times j)_!} & H_{*-d}(LM \times LM \times LM) \\
 \tilde{j}_* \uparrow & & (j_1)_* \uparrow & & (j \times 1)_* \uparrow \\
 H_{*-2d}(LM \times_M LM) & \xleftarrow{(\iota_1)_*} & H_{*-2d}(LM \times_M LM \times_M LM) & \xleftarrow{(j_2)_!} & H_{*-d}(LM \times_M LM \times LM) \\
 \tilde{\iota}_! = \iota_! \uparrow & & (\iota_2)_! \uparrow & & (\iota \times 1)_! = \iota_! \times 1 \uparrow \\
 H_{*-d}(LM) & \xleftarrow{\iota_*} & H_{*-d}(LM \times_M LM) & \xleftarrow[\tilde{j}_! = (-1)^d j_!]{\tilde{j}_!} & H_*(LM \times LM)
 \end{array}$$

where the transfer maps along the perimeter has been identified as shown. Using similar methods, all the squares commute except the top right one which commutes up to $(-1)^d$. With this information, following the diagram via top right corner gives $(-1)^{d+d|a|}\Psi(a) \cdot b$, and following the diagram via the bottom left corner gives $(-1)^{d+d(|a|-d)}\Psi(a \cdot b)$. Since the entire diagram commutes up to $(-1)^d$, we obtain the identity $\Psi(a \cdot b) = \Psi(a) \cdot b$. This completes the proof. \square

Note that in the same diagram of fibrations, if we consider an induced homology diagram with transfers flowing from the top right corner to the bottom left corner, or a diagram flowing from the bottom left corner to the top right corner, we obtain homotopy theoretic proofs of associativity of the loop product [CJ] and the coassociativity of the loop coproduct.

Next we show that Ψ is symmetric. Let $T : LM \times LM \rightarrow LM \times LM$ be the switching map.

Proposition 2.3. *The loop coproduct is symmetric in the sense that*

$$T_*(\Psi(a)) = \Psi(a)$$

for any $a \in H_*(LM)$.

Proof. We consider the following commutative diagram:

$$\begin{array}{ccccc}
 LM & \xleftarrow{\iota} & LM \times_M LM & \xrightarrow{j} & LM \times LM \\
 R_{\frac{1}{2}} \downarrow & & T \downarrow & & T \downarrow \\
 LM & \xleftarrow{\iota} & LM \times_M LM & \xrightarrow{j} & LM \times LM
 \end{array}$$

Here, as before, we identify SM with $LM \times_M LM$, and $R_{\frac{1}{2}}$ is the rotation of loops by $\frac{1}{2}$, that is $R_{\frac{1}{2}}(\gamma)(t) = \gamma(t + \frac{1}{2})$. The left square commutes because $R_{\frac{1}{2}} \circ \iota(\gamma, \eta) = R_{\frac{1}{2}}(\gamma \cdot \eta) = \eta \cdot \gamma = \iota \circ T(\gamma, \eta)$. The Thom class for the embedding ι is given by $\tilde{u} = p^*(e_M)$. Since $R_{\frac{1}{2}} \simeq 1$, we have $R_{\frac{1}{2}}^*(\tilde{u}) = \tilde{u}$. Thus the Thom classes for two ι 's are compatible and we have $T_* \circ \iota_! = \iota_! \circ R_{\frac{1}{2}*} = \iota_!$. Thus the above commutative diagram implies $T_*(\Psi(a)) = T_* \circ j_* \circ \iota_!(a) = j_* \circ T_* \circ \iota_!(a) = j_* \circ \iota_!(a) = \Psi(a)$. \square

The loop coproduct behaves well with respect to cap products with cohomology classes in $H^*(LM)$ arising from $\alpha \in H^*(M)$. Let $p : LM \rightarrow M$ be the base point map. For the evaluation map $e = p \circ \Delta : S^1 \times LM \rightarrow M$, let $e^*(\alpha) = 1 \times p^*(\alpha) + \{S^1\} \times \Delta(p^*(\alpha))$, where $\{S^1\}$ is the fundamental cohomology class for S^1 .

Theorem 2.4. *Let $\alpha \in H^*(M)$ and $b \in H_*(LM)$.*

(1) *The cap product with $p^*(\alpha)$ satisfies Frobenius compatibility with respect to the loop coproduct:*

$$(2-4) \quad \Psi(p^*(\alpha) \cap b) = (-1)^{d|\alpha|} (p^*(\alpha) \times 1) \cap \Psi(b) = (-1)^{d|\alpha|} (1 \times p^*(\alpha)) \cap \Psi(b).$$

(2) *The cap product with $\Delta(p^*(\alpha))$ behaves as a coderivation with respect to the loop coproduct:*

$$(2-5) \quad \Psi(\Delta(p^*(\alpha)) \cap b) = (-1)^{d(|\alpha|-1)} [\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta(p^*(\alpha))] \cap \Psi(b).$$

Proof. From the definition of the loop coproduct and a property (2-2) of the transfer $\iota_!$, we have $\Psi(p^*(\alpha) \cap b) = j_* \circ \iota_! (p^*(\alpha) \cap b) = (-1)^{d|\alpha|} j_* (\iota^* p^*(\alpha) \cap \iota_!(b))$. To understand $\iota^* p^*(\alpha)$, we consider the following commutative diagram.

$$\begin{array}{ccccc} LM & \xleftarrow{\iota} & LM \times_M LM & \xrightarrow{j} & LM \times LM \\ p \times p' \downarrow & & q \downarrow & & p \times p \downarrow \\ M \times M & \xleftarrow{\phi} & M & \xrightarrow{\phi} & M \times M \\ \pi_1 \downarrow & & \parallel & & \pi_i \downarrow \\ M & \xlongequal{\quad} & M & \xlongequal{\quad} & M \end{array}$$

where $p'(\gamma) = \gamma(\frac{1}{2})$, and π_i for $i = 1, 2$ is the projection onto the i th factor. From the diagram, we have $\iota^* p^*(\alpha) = q^*(\alpha) = j^*(p \times p)^* \pi_i^*(\alpha)$, which is equal, for $i = 1, 2$, to $j^*(p^*(\alpha) \times 1)$ and to $j^*(1 \times p^*(\alpha))$. For $i = 1$ case,

$$(-1)^{d|\alpha|} \Psi(p^*(\alpha) \cap b) = j_* (j^*(p^*(\alpha) \times 1) \cap \iota_!(b)) = (p^*(\alpha) \times 1) \cap j_* \iota_!(b) = (p^*(\alpha) \times 1) \cap \Psi(b).$$

Similarly, for the case $i = 2$, we obtain $(-1)^{d|\alpha|} \Psi(p^*(\alpha) \cap b) = (1 \times p^*(\alpha)) \cap \Psi(b)$.

For (2), first we note that

$$\Psi(\Delta(p^*(\alpha)) \cap b) = j_* \iota_! (\Delta(p^*(\alpha)) \cap b) = (-1)^{d(|\alpha|-1)} j_* (\iota^* \Delta(p^*(\alpha)) \cap \iota_!(b)).$$

We need to understand $\iota^* (\Delta(p^*(\alpha)))$. For this purpose, we introduce some notations. Let $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. Let $r : S^1 = I/\partial I \rightarrow I/\{0, \frac{1}{2}, 1\} = S^1_1 \vee S^1_2$, where $S^1_i = I_i/\partial I_i$ for $i = 1, 2$, be an identification map. Let $\iota_i : S^1_i \rightarrow S^1_1 \vee S^1_2$ be the inclusion map for $i = 1, 2$. We consider the following commutative diagram.

$$\begin{array}{ccccc} S^1_i \times (LM \times_M LM) & \xrightarrow{1 \times j} & S^1_i \times (LM \times LM) \\ \iota_i \times 1 \downarrow & & 1 \times \pi_i \downarrow \\ S^1 \times (LM \times_M LM) & \xrightarrow{r \times 1} & (S^1_1 \vee S^1_2) \times (LM \times_M LM) & & S^1_i \times LM \cong S^1 \times LM \\ 1 \times \iota \downarrow & & e' \downarrow & & e \downarrow \\ S^1 \times LM & \xrightarrow{e} & M & \xlongequal{\quad} & M \end{array}$$

where $e'(t, \gamma, \eta)$ is given by $\gamma(2t)$ for $0 \leq t \leq \frac{1}{2}$, and $\eta(2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. Let $e'^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{S^1_1\} \times \Delta_1(\alpha) + \{S^1_2\} \times \Delta_2(\alpha)$, where the first term is due to a fact that e' restricted

to $\{0\} \times (LM \times_M LM)$ is given by $p \circ \iota$. Since $e^*(\alpha) = 1 \times p^*(\alpha) + \{S^1\} \times \Delta p^*(\alpha)$ and $r^*(\{S_i^1\}) = \{S^1\}$ for $i = 1, 2$, the commutativity of the left bottom square implies that

$$\iota^* \Delta(p^*(\alpha)) = \Delta_1(\alpha) + \Delta_2(\alpha).$$

We need to identify $\Delta_i(\alpha)$ for $i = 1, 2$. The commutativity of the right square implies that, for $i = 1$, $(1 \times j)^*(1 \times \pi_1)^* e^*(\alpha) = 1 \times j^*(p^*(\alpha) \times 1) + \{S^1\} \times j^*(\Delta(p^*(\alpha)) \times 1)$ is equal to $(\iota_1 \times 1)^* e'^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{S^1\} \times \Delta_1(\alpha)$. Hence $\Delta_1(\alpha) = j^*(\Delta(p^*(\alpha)) \times 1)$. Similarly, the $i = 2$ case implies that $\Delta_2(\alpha) = j^*(1 \times \Delta(p^*(\alpha)))$. Combining the above calculations, we have

$$\begin{aligned} \Psi(\Delta(p^*(\alpha)) \cap b) &= (-1)^{d(|\alpha|-1)} (\iota^* \Delta(p^*(\alpha)) \cap \iota_!(b)) \\ &= (-1)^{d(|\alpha|-1)} j_* (j^*(\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta(p^*(\alpha))) \cap \iota_!(b)) \\ &= (-1)^{d(|\alpha|-1)} [\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta(p^*(\alpha))] \cap \Psi(b). \end{aligned}$$

This proves the coderivation property. \square

§3 Properties of the loop coproduct and their consequences

So far we have proved various algebraic properties of the loop coproduct. These properties turn out to be strong enough to force the loop coproduct to be given by a very simple formula, given in the next theorem. Let $s : M \rightarrow LM$ be the constant loop map given by $s(x) = c_x$, where c_x is the constant loop at $x \in M$. Recall that we assume that M is connected with base point x_0 , and let c_0 be the constant loop at the base point.

The connected components of LM are in 1:1 correspondence to the set of free homotopy classes of loops $[S^1.M]$, which is in 1:1 correspondence with conjugacy classes of $\pi_1(M)$. Let

$$LM = (LM)_{[1]} \cup \bigcup_{[\alpha] \neq [1]} (LM)_{[\alpha]},$$

be the decomposition of LM into its components, where $[\alpha]$'s are conjugacy classes in $\pi_1(M)$.

Theorem 3.1. *Let M be a connected oriented closed smooth d -manifold.*

(1) *Let $p \geq 0$ and let $a_1, a_2, \dots, a_p \in H_*(LM)$. The loop coproduct on the loop product of these elements is given by the following formula, for each $0 \leq \ell \leq p$.*

$$(3-1) \quad \Psi(a_1 a_2 \cdots a_p) = \chi(M) ([c_0] \cdot a_1 \cdot a_2 \cdots a_\ell) \otimes ([c_0] \cdot a_{\ell+1} \cdots a_p) \in H_*(LM) \otimes H_*(LM).$$

In particular, for the unit $1 = s_([M]) \in H_d(LM) = \mathbb{H}_0(LM)$ of the loop homology algebra, its coproduct is given by*

$$(3-2) \quad \Psi(1) = \chi(M) [c_0] \otimes [c_0] \in H_0(LM) \otimes H_0(LM) \cong H_0(LM \times LM).$$

When $p = 1$, the formula for $a \in H_(LM)$ for $\ell = 0, 1$ becomes*

$$(3-3) \quad \Psi(a) = \chi(M) ([c_0] \cdot a) \otimes [c_0] = \chi(M) [c_0] \otimes ([c_0] \cdot a).$$

(2) *If $|a| \neq d$, then $\Psi(a) = 0$. If $|a| = d$, then $\Psi(a) = n[c_0] \otimes [c_0]$ for some $n \in \mathbb{Z}$. Thus, $\text{Im } \Psi = \mathbb{Z}[c_0] \otimes [c_0]$.*

(3) *Suppose $a \in H_d((LM)_{[\alpha]})$ be a degree d homology class in $[\alpha]$ -component of LM . If $[\alpha] \neq [1]$, then $\Psi(a) = 0$.*

(4) Suppose $a \in H_d((LM)_{[1]})$, and suppose it is of the form $a = ks_*([M]) + (\text{decomposables})$ in the loop algebra $H_*(LM)$ for some $k \in \mathbb{Z}$, then

$$\Psi(a) = k\chi(M)[c_0] \otimes [c_0].$$

Proof. First, we prove the formula for $\Psi(1)$. Since $1 = s_*([M])$ has degree d , and $\iota_!$ decreases degree by d , we have $\iota_!(1) \in H_0(LM \times_M LM)$. Since M is connected, connected components of LM are in 1:1 correspondence with conjugacy classes of $\pi_1(M)$. Let L_0M be the component consisting of contractible loops so that $c_0 \in L_0M$. Note that $L_0M \times_M L_0M$ is also connected, and $H_0(L_0M \times_M L_0M) \cong \mathbb{Z}$ is generated by $[(c_0, c_0)]$. So we may write $\iota_!(1) = m[(c_0, c_0)]$ for some $m \in \mathbb{Z}$. Since $\iota_* : H_0(L_0M \times_M L_0M) \rightarrow H_0(L_0M)$ is an isomorphism with $\iota_*([(c_0, c_0)]) = [c_0]$, and since (2-1) implies $\iota_*\iota_!(1) = p^*(e_M) \cap s_*([M]) = s_*(e_M \cap [M]) = \chi(M)[c_0]$, we have $\iota_!(1) = \chi(M)[(c_0, c_0)]$. Hence $\Psi(1) = j_*\iota_!(1) = \chi(M)[c_0] \otimes [c_0]$.

For $a_1, a_2, \dots, a_p \in H_*(LM)$ and for $0 \leq \ell \leq p$, the Frobenius compatibility (2-3) implies

$$\begin{aligned} \Psi(a_1 \cdot a_2 \cdots a_p) &= (-1)^{d(|a_1| + \cdots + |a_\ell| - d\ell)} (a_1 \cdots a_\ell) \cdot \Psi(1) \cdot a_{\ell+1} \cdots a_p \\ &= (-1)^{d(|a_1| + \cdots + |a_\ell| - d\ell)} \chi(M) a_1 \cdots a_\ell \cdot [c_0] \otimes [c_0] \cdot a_{\ell+1} \cdots a_p \\ &= \chi(M) ([c_0] \cdot a_1 \cdots a_\ell) \otimes ([c_0] \cdot a_{\ell+1} \cdots a_p). \end{aligned}$$

Here, we used the graded commutativity in the loop homology algebra given by

$$a \cdot b = (-1)^{(|a|-d)(|b|-d)} b \cdot a, \quad a, b \in H_*(LM).$$

When $p = 1$, we get the formula for $\Psi(a)$ given in (3-3). Note that the formula is compatible with the symmetry formula $T_*\Psi(a) = \Psi(a)$ in Proposition 2.3. Note also that our formula tells us that the image of Ψ is contained in the tensor product $H_*(LM) \otimes H_*(LM) \subset H_*(LM \times LM)$, essentially because $\Psi(1)$ is by (3-2).

(2) From the formula (3-3), the value $\Psi(a)$ must be an integral multiple of $[c_0] \otimes [c_0] \in H_0(LM) \otimes H_0(LM)$. Since Ψ lowers degree by d , if $|a| \neq d$, we must have $\Psi(a) = 0$.

(3) Let $a \in H_d((LM)_{[\alpha]})$. We show that if $\Psi(a) \neq 0$, then $[\alpha] = [1]$. By (2), $\Psi(a)$ must be of the form $n[c_0] \otimes [c_0]$ for some $n \in \mathbb{Z}$. Comparing with (3-3), if $\Psi(a) \neq 0$, then $[c_0] \cdot a = k[c_0]$ for some $k \neq 0$, which is a homology class of finite union of contractible loops. Thus a must be represented by a cycle in the space of contractible loops $(LM)_{[1]}$. Hence we have $[\alpha] = [1]$.

(4) By (2), if $|a| \neq d$, we must have $\Psi(a) = 0$, which is equivalent to

$$(3-4) \quad \chi(M)[c_0] \cdot a = 0, \quad a \in H_*(LM) \text{ with } |a| \neq d.$$

Now suppose $|a| = d$ and a is decomposable of the form $a = \sum_i b'_i \cdot b''_i$ with $|b'_i| \neq d$ for all i , then $\Psi(a) = \sum_i \chi(M)[c_0] \cdot b'_i \otimes [c_0] \cdot b''_i = 0$ by (3-1). Thus, if a is of the form $a = ks_*([M]) + (\text{decomposables})$, then $\Psi(a) = \Psi(ks_*([M])) = k\chi(M)[c_0] \otimes [c_0]$. \square

Implications of Theorem 3.1 are rather striking. First, we start with straightforward corollaries whose proofs are obvious.

Corollary 3.2. *Let M be a connected closed oriented smooth manifold. If its Euler characteristic is zero, then the loop coproduct vanishes identically.*

In particular, if M is odd dimensional, then the loop coproduct vanishes identically.

For example, the loop coproduct vanishes in $H_*(LS^{2n+1})$. The above Corollary 3.2 was also observed in [S].

Next, we examine torsion elements in loop homology.

Corollary 3.3. *Assume that $\chi(M) \neq 0$ for a connected closed oriented smooth d -manifold M . For any element $a \in H_*(LM)$ with $|a| \neq d$, the element $[c_0] \cdot a$ is either 0 or a torsion element of order a divisor of $\chi(M)$.*

Proof. In the proof of Theorem 3.1, we noted that $\chi(M)[c_0] \cdot a = 0$ if $|a| \neq d$ in (3-4). Since $\chi(M) \neq 0$, the conclusion follows. \square

When $|a| = d$, the element $[c_0] \cdot a$ lies in $H_0(LM)$, so it is either 0 or torsion free.

Let $\iota : \Omega M \rightarrow LM$ be the inclusion map from the based loop space to the free loop space. Recall that we have an algebra map

$$\iota_! : H_{*+d}(LM) \rightarrow H_*(\Omega M)$$

from the loop algebra to the Pontrjagin ring, where $d = \dim M$.

Corollary 3.4. *Suppose $\chi(M) \neq 0$ for a closed oriented smooth d -manifold M . Then for $p \neq 0$, the image of the composition*

$$\iota_* \circ \iota_! : H_{p+d}(LM) \rightarrow H_p(\Omega M) \rightarrow H_p(LM)$$

consists entirely of torsion elements of order a divisor of $\chi(M)$.

Proof. Since $\iota_* \circ \iota_!(a) = [c_0] \cdot a$ for $a \in H_*(LM)$, the assertion follows from Corollary 3.3. \square

Next, we show that similar statements hold for loop bracket products of the form $\{[c_0], a\}$ for $a \in H_*(LM)$.

Corollary 3.5. *Suppose $\chi(M) \neq 0$ for a closed connected oriented smooth d -manifold M , and let $a \in H_*(LM)$.*

- (1) *If $|a| \neq d, d-1$, then the element $\{[c_0], a\}$ is either 0 or a torsion element of order a divisor of $\chi(M)$.*
- (2) *Suppose further M is simply connected. Then if $|a| \neq d-1$, then the element $\{[c_0], a\}$ is either 0 or a torsion element of order a divisor of $\chi(M)$.*

Proof. Since $\chi(M) \neq 0$, M is even dimensional. The BV-identity multiplied by $\chi(M)$ gives

$$\Delta(\chi(M)[c_0] \cdot a) = \chi(M)\Delta([c_0]) \cdot a + \chi(M)[c_0] \cdot \Delta(a) + \chi(M)\{[c_0], a\}.$$

If $|a| \neq d, d-1$, then by Corollary 3.3, we have $\chi(M)[c_0] \cdot a = 0$ and $\chi(M)[c_0] \cdot \Delta(a) = 0$. Since S^1 action on M is trivial, we have $\Delta([c_0]) = 0$. Thus $\chi(M)\{[c_0], a\} = 0$, and the conclusion of (1) follows.

For (2), when $|a| = d$, the element $\Delta(a)$ has degree $d+1$. By Corollary 3.3, $\chi(M)[c_0] \cdot \Delta(a) = 0$. If M is simply connected, LM has a single component L_0M and so $[c_0] \cdot a \in H_0(LM) \cong \mathbb{Z}$ generated by $[c_0]$. Since $\Delta([c_0]) = 0$, we have $\Delta([c_0] \cdot a) = 0$. Hence $\chi(M)\{[c_0], a\} = 0$, from which the conclusion follows. \square

When $|a| = d-1$, since $\{[c_0], a\} \in H_0(LM)$, this element is either 0 or torsion free. To see what happens when M is not simply connected, for each conjugacy class $[g]$ of $\pi_1(M)$ we choose a loop γ_g in M belonging to $[g]$. When $|a| = d$, the element $[c_0] \cdot a$ is a linear combination of classes $[\gamma_g] \in H_0(LM)$. Since $\Delta([\gamma_g]) \in H_1(L_{[g]}M)$ can be nonzero, the simple connectivity assumption is needed in (2) of Corollary 3.5.

Example 3.6. We can verify Corollary 3.3 in actual examples. In [CJY], the loop homology algebra for LS^{2n} and LCP^n are computed. Their computation shows

$$\begin{aligned}\mathbb{H}_*(LS^{2n}) &\cong \Lambda(b) \otimes \mathbb{Z}[a, v]/(a^2, ab, 2av), & b \in \mathbb{H}_{-1}, a \in \mathbb{H}_{-2n}, v \in \mathbb{H}_{4n-2}, \\ \mathbb{H}_*(LCP^n) &\cong \Lambda(w) \otimes \mathbb{Z}[c, u]/(c^{n+1}, (n+1)c^n u, wc^n), & w \in \mathbb{H}_{-1}, c \in \mathbb{H}_{-2}, u \in \mathbb{H}_{2n}.\end{aligned}$$

For $\mathbb{H}_*(LS^{2n})$, we have $[c_0] = a$ and $\chi(S^{2n}) = 2$. By the above computation, we can easily see that $\chi(S^{2n})[c_0] \cdot x = 2a \cdot x = 0$ for all $x \in \mathbb{H}_*(LS^{2n})$ not in \mathbb{H}_0 . For $\mathbb{H}_*(LCP^n)$, we have $[c_0] = c^n$ and $\chi(CP^n) = n+1$. Again we can easily see that the identity $\chi(CP^n)[c_0] \cdot y = (n+1)c^n \cdot y = 0$ for all y not in \mathbb{H}_0 .

We discuss two final related results. The first one concerns an analogue of the BV identity for the loop coproduct. The BV identity can be understood by saying that the failure of the commutativity of the following diagram is the loop bracket:

$$\begin{array}{ccc} H_*(LM) \otimes H_*(LM) & \xrightarrow{\text{loop product}} & H_*(LM) \\ \Delta \otimes 1 + 1 \otimes \Delta \downarrow & & \Delta \downarrow \\ H_*(LM) \otimes H_*(LM) & \xrightarrow{\text{loop product}} & H_*(LM). \end{array}$$

We ask a similar question for the loop coproduct. Does the following diagram commute? If not, what is the measure of the failure of the commutativity?

$$\begin{array}{ccc} H_*(LM) & \xrightarrow{\Psi} & H_*(LM \times LM) \\ \Delta \downarrow & & \Delta \times 1 + 1 \times \Delta \downarrow \\ H_*(LM) & \xrightarrow{\Psi} & H_*(LM \times LM) \end{array}$$

Unfortunately, things turn out to be rather trivial for the loop coproduct.

Proposition 3.7. *For every $a \in H_*(LM)$, the identity $(\Delta \times 1 + 1 \times \Delta)\Psi(a) = 0$ holds.*

Proof. For $a \in H_*(LM)$, by (2) of Theorem 3.1, $\Psi(a) \in \mathbb{Z}[c_0] \otimes [c_0] \subset H_0(LM \times LM)$. Since $\Delta([c_0]) = 0$, the above identity holds. \square

For the second result, recall that the loop product and the loop coproduct satisfy Frobenius compatibility (Theorem 2.2). We ask a similar question. What is the compatibility relation for the loop bracket and the loop coproduct? The result turns out to be trivial when one of the elements is from $H_*(M)$.

Proposition 3.8. *Let M be as before with $\chi(M) \neq 0$. Suppose $a \in H_*(M)$. Then for any $b \in H_*(LM)$, we have $\Psi(\{a, b\}) = 0$.*

Proof. Let $\alpha \in H^*(M)$ be the cohomology class dual to a . Since $\Delta\alpha \cap b = (-1)^{|\alpha|}\{a, b\}$ (see [T1]), using the coderivation property of the cap product with respect to the loop coproduct (2-5),

$$\begin{aligned}\Psi(\{a, b\}) &= (-1)^{|\alpha|+(|\alpha|-1)d}(\Delta\alpha \times 1 + 1 \times \Delta\alpha) \cap \Psi(b) \\ &= (-1)^{|\alpha|+(|\alpha|-1)d}[\chi(M)(\Delta\alpha \cap ([c_0] \cdot b)) \otimes [c_0] + \chi(M)[c_0] \otimes (\Delta\alpha \cap ([c_0] \cdot b))].\end{aligned}$$

Since the loop bracket behaves as a derivation in each variable, and $\{a, [c_0]\} = 0$ for $a \in H_*(M)$, we have $\Delta\alpha \cap ([c_0] \cdot b) = (-1)^{|\alpha|}\{a, [c_0] \cdot b\} = (-1)^{|\alpha|+(|\alpha|+1)d}[c_0] \cdot \{a, b\}$. The above identity then becomes

$$\Psi(\{a, b\}) = \chi(M)([c_0] \cdot \{a, b\}) \otimes [c_0] + \chi(M)[c_0] \otimes ([c_0] \cdot \{a, b\}) = \Psi(\{a, b\}) + \Psi(\{a, b\}),$$

using (3-3). Hence $\Psi(\{a, b\}) = 0$. \square

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